MINOR RESEARCH PROJECT
FINAL REPORT

1. **Introduction:** Fractional calculus has been known since the 17th century. Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The basic Mathematical ideas of fractional calculus were developed long time by the Mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and others. One owes to Leibniz in a letter to L'Hospital, dated September 30, 1695, the exact birthday of the fractional calculus and the idea of the fractional derivative. During three Centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians. In the few last decades, however, fractional differentiation proved very useful in various fields of applied sciences and engineering. The class of fractional operator equations of various types plays very important role not only in mathematics but also in physics, chemistry, biology, economics, signal and image processing, calculus of variations, control theory, electrochemistry, viscoelasticity, feedback amplifier and electrical circuits. Fractional order integral and differential equations play a very important role in many applications of real world problem.
2. **Importance of the study of Nonlinear integral and differential equations:** It is common knowledge that the several natural and physical problems are nonlinear in nature and so governed by nonlinear equations like differential, integral etc. Since nonlinear, non-deterministic nature of phenomena in the general area of the biological, engineering and physical sciences, the mathematical description of such phenomena result in the differential, integral and integro-differential equations also, many processes or phenomena in physics and biology are not continuous and involve jumps or discontinuous. The coefficients or parameters that have an important role in the natural processes are called random parameters. Hence when we talk about some parameters or coefficients, the random analysis of the random equations is evident. Therefore the random equations have been studied in the literature, since long time, by various Mathematicians all over the world. Thus the study of a natural or physical phenomenon with the help of random models or equations forms an important branch of the analysis. Therefore some of these problems may be formulated as nonlinear equations involving discontinuous terms. Therefore it is of interest to study the nonlinear differential, integral and integro-differential equations and there applications to day to day problems of the life.
3. **The objective of the Present Work:** The main interest in the present investigation to discuss the nonlinear differential and integral equations of fractional order for their different behavior of the solutions and discuss some applications of the nonlinear differential and integral equation studied in the present work of some concrete problems of other areas of Mathematics.
4. **Methodology:** There are different approaches or methods for dealing with nonlinear differential and integral equations
   (a) differential or integral inequality method
   (b) comparison method
   (c) approximation method and
   (d) Fixed point method etc.

Fixed point theoretical method is the most powerful one. There are several fixed point theorems which are useful in applications to nonlinear differential, integral equation. Methodology for dealing with the differential equations consists of the following steps. In the first step, convert the given differential equation in to the integral equation, then second step convert so obtained integral equation in to the equivalent operator equations into some suitable function spaces. In the third step, we apply some suitable fixed point theorem to the operator equations to yield the existence of the solution.

5.Operator theory: Let *X* be a Banach space. Then a mapping $T : X \to X$ is called a operator and functional equation $T(x) = x$ is called an operator equation. The problem of existence of the solutions to such operator equations in abstract spaces is of great interest since long time. A mathematical statement asserting the existence of solutions to the operator equations is called the fixed point theorem and the solutions themselves are called the fixed points of *T* in *X*. Such fixed-point results are obtained by imposing certain condition either on *f* or on *X* or on both *f* and *X*. These fixed point results are useful for proving the existence theorems for various nonlinear problems of the universe modelled on differential and integral equations, called the dynamical systems. Below we discuss some of the results in this direction.

5.1. Fixed point theory: As has been stated that the topic of fixed point theory provides powerful tools in the study of nonlinear equaions, it is classified into three main categories, viz, algebraic, geometrical and topological fixed point theory. The above classification is not rigid, but based on the major hypotheses involved in the fixed-point theorems in abstract spaces. In the following, we state some basic theorems from the above categories useful for applications to allied areas of mathematics.

Definition 5.1.1 : Let *X* be a Banach space. A mapping $T : X \to X$ is called Lipschitz if there is a constant $\alpha > 0$ such that $\|Tx - Ty\| \le \alpha \|x - y\|$ for all $x, y \in X$. If $\alpha < 1$, then *T* is called a contraction on *X* with the contraction constant α.

Theorem 5.1.1 : *Let T be a contraction on a Banach space X into itself. Then T has a unique fixed point.*

Before stating a basic topological fixed-point theorem, we need the following preliminaries.

Let *X* be a Banach space and Let $T : X \to X$, *T* is called compact if $\overline{T(X)}$ is a compact subset of *X*. *T* is called totally bounded if $\overline{T(S)}$ is a compact subset of *X* for every bounded subset *S* of *X*. *T* is called completely continuous if it is continuous and totally bounded operator on *X*. It is clear that every compact operator is totally bounded but the converse need not be true.

Theorem 5.1.2: Let S be a closed, convex and bounded subset of a Banach space X and Let $T : S \to X$ be a completely continuous operator. Then T has a fixed point.

Definition 5.1.2: Let *X* be an ordered Banach space. A mapping $Q : X \to X$ is called nondecreasing if $Qx \le Qy$ for all $x, y \in X$ for which $x \le y$.

5.2. Hybrid fixed point theory: A fixed point theorem involving the mixed hypotheses from algebra, geometry and topology is called a hybrid fixed point theorem in abstract spaces. The study of hybrid fixed point theory is initiated by a Russian mathematician Krasnoselskii in the year 1964 and proved a fixed point theorem which combines the Banach fixed point theorem together with the Schauder fixed point theorem on subsets of a Banach space.

Theorem 5.2.1 : (Krasnoselskii) Let S be a closed convex and bounded subset of the Banach space X. Let $A: X \to X$ and $B: S \to X$ be two operators such that

(a) A is a contraction,

(b) B is completely continuous and

(c) $Ax + By \in S$ for all $x, y \in S$.

Then the operator equation $Ax + Bx = x$ has a solution in S.

Other two hybrid fixed point theorems involving two and three operators in a Banach algebras are as follows.

Theorem 5.2.2 : (Dhage) Let S be a closed convex and bounded subset of a Banach algebra X and let $A: X \to X$ and $B: S \to X$ be two operators such that

(a) A is Lipschitz with the Lipschitz constant α,

(b) B is completely continuous, and

(c) $AxBy \in S$ for all $x, y \in S$.

Then the operator equation $AxBx = x$ has a solution in S, whenever $\alpha M < 1$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

Theorem 5.2.3: (Dhag**e**) Let S be a closed, convex and bounded subset of the Banach algebra X and Let $A, C: X \to X$ and $B: S \to X$ be three operators satisfying

(a) A and C are Lipschitz with the Lipschitz constants α and β respectively,

(b) B is a completely continuous, and

(c) $AxBy + Cx \in S$ for all $x, y \in S$.

Then the operator equation $AxBx + Cx = x$ has a solution in S, whenever $\alpha M + \beta < 1$, where $M = \| B(S) \| = \sup\{\| Bx \| : x \in X\}$.

We use some variants of the above two fixed point theorems in Banach algebras for proving the existence results for differential or integral equations in Banach algebras.

6. **Random Operator Theory:** Let (Ω, A, μ) denote a measure space, X a separable Banach Space and Y another Banach space. For a fixed $\omega \in \Omega$, $F(\omega)$ denotes a deterministic operator from X into Y.

Definition: A mapping $T : \Omega \times X \to Y$ is said to be a random operator if for given $B \in \beta_Y$, the set $\{\omega \in \Omega / \mathrm{T}(\omega) x \in B\} \in A$ for all $x \in X$.

The above definition simply says that $T(\omega)$ is a random operator if $T(\omega)x = y(\omega)$,say is a Y-valued random variable for each $x \in X$

6.1.Random fixed point theory: Random fixed point theory is an important topic of the nonlinear probabilistic functional analysis which provides some powerful tools for the study of nonlinear random equations. The systematic study of the nonlinear random equations using the random fixed point theory was first initiated by Prague. In the present work, we have used some known random fixed point theorems for nonlinear random operators in the study of fractional order nonlinear random integral and differential equations.

Definition 6.1.1.: Let $T : \Omega \times X \to X$ be a random operator. A random variable $\xi \in X$ is called a random fixed point of the random operator $T(\omega)$ if $T(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$

and any mathematical statement asserting the existence of a random fixed point is called a random fixed point theorem.
There are only basic three approaches for dealing with the nonlinear random differential and integral equations, namely,

(a) Geometric method ,
(b) Topological method ,
(c) Algebric methods.

Below we discuss topological method for different aspects of the random solutions.

Topological random fixed point theory: The topological random fixed point theory for completely continuous random operators is generally used in the study of random solutions for random differential and integral equations. We recall definitions from Banach spaces.

Let $T:\Omega\times X\to X$ be a random operator. $T(\omega)$ is called compact if $\overline{T(\omega)(X)}$ is a compact subset of X for each $\omega\in\Omega$. $T(\omega)$ is called totally bounded if $T(\omega)(B)$ is a totally bounded subset of X for each $\omega\in\Omega$, where B is a bounded subset of X. Finally, $T(\omega)$ is called completely continuous if it is continuous and totally bounded random operator in X. Notice that every compact random operator is totally bounded random operator on X, but the converse is not necessarily true. The applicable random fixed point theorem is.

Theorem 6.1.1 :(Dhage). Let S be a closed convex and bounded subset of the Separable Banach space X and let $A(\omega),B(\omega):\Omega\times S\to X$ be two operators satisfying for each $\omega\in\Omega$:

(a) $A(\omega)$ is Lipschitzician ,
(b) $B(\omega)$ is completely continuous, and
(c) $A(\omega)xB(\omega)x\in S$ for all $x\in S$,and

Then the operator equation $AxBx=x$ has a random solution whenever $M(\omega)\phi_\omega(r)<r, r>0$ for each $\omega\in\Omega$ where $M(\omega)=\|B(\omega)(S)\|=\sup\{\|Bx\|:x\in S\}$

Theorem 6.1.2: $(Dhage)$: Let S be a closed convex and bounded subset of the Separable Banach space X and let $A,B,C:\Omega\times S\to X$ be three random operators satisfying for each $\omega\in\Omega$:

(a) $A(\omega)$ and $C(\omega)$ are D-Lipschitzicians with D-functions $\phi_A(\omega)$ and $\phi_C(\omega)$ respectively,
(b) $B(\omega)$ is completely continuous, and
(c) $A(\omega)xB(\omega)x+C(\omega)x\in S$ for $x\in S$.

Then the operator equation $A(\omega)xB(\omega)x+C(\omega)x=x$ has a random solution whenever $M(\omega)\phi_A(\omega)(r)+\phi_C(\omega)(r)<r, r>0$, for all $\omega\in\Omega$

Where $M(\omega) = \|B(\omega)(S)\| = \sup\{\|B(\omega)x\| : x \in S\}$

An interesting corollary to Theorem (2.3) in its applicable form is

Corollary 6.1.3: $(Dhage)$: Let S be a closed convex and bounded subset of the Separable Banach space X and let $A, B, C : \Omega \times S \to X$ be three random operators satisfying for each $\omega \in \Omega$:

(a) $A(\omega)$ and $C(\omega)$ are D-Lipschitzicians with the lipschitz constant $\alpha(\omega)$ and $\beta(\omega)$ respectively,

(b) $B(\omega)$ is continuous and compact , and

(c) $A(\omega)xB(\omega)x + C(\omega)x \in S$ for each $x \in S$.

Then the operator equation $A(\omega)xB(\omega)x + C(\omega)x = x$ has a random solution and the set of such random solutions is compact, whenever $\alpha(\omega)M(\omega) + \beta(\omega) < 1$, for all $\omega \in \Omega$ where

$M(\omega) = \|B(\omega)(S)\| = \sup\{\|B(\omega)x\| : x \in S\}$.

7. Existence Result: The study of nonlinear differential equations is initiated by Dhage and Regan in the year 2001 and developed further in a series of papers by first authors and others. It is well known that the hybrid fixed point theorems are useful for proving the existence the solutions for various classes of differential and integral equations. In the present work; we shall discuss nonlinear integral equation in Banach algebras for existence as well as for existence of the locally attractive solutions. We study the existence results of locally attractive solutions of the following nonlinear integral and differential equations of fractional order.

(a)Let R denote the real numbers whereas R_+ be the set of nonnegative numbers consider the nonlinear quadratic functional integral equation of fractional order:

$$x(t) = f\left(t, x(\alpha(t))\right)\left[q(t) + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g\left(s, x(\gamma(s))\right) ds\right] \quad \text{for all} \quad t \in R_+, \quad \beta \in (0,1)$$

Where $q : R_+ \to R$, $f : R_+ \times R \to R$, $g(t,x) = g : R_+ \times R \to R$ and $\alpha, \gamma : R_+ \to R_+$.

By a Solution of the FIE we mean a function $x \in BC(R_+, R)$ that satisfies on R_+

Where $BC(R_+, R)$ is the space of continuous and bounded real-valued functions defined on R_+

(b) Let R denote the real line and R_+ be the set of nonnegative numbers, i.e. $R_+ = [0, \infty) \subset R$ denote the $BC(R_+, R)$ class of bounded continuous real valued functions defined on R_+ Given a measurable spac (Ω, A) and for given function $x : \Omega \to BC(R_+, R)$ .Consider the fractional order nonlinear functional random integral equation (in short FNRIE) of mixed type

$$x(t,\omega)= f\left(t,x(\theta(t),\omega),\omega\right)\left[q(t,\omega)+\frac{1}{\Gamma(\alpha)}\int_0^{\mu(t)}\frac{g\left(s,x(\eta(s),\omega),\omega\right)}{(t-s)^{1-\alpha}}ds\right]$$

for all $t\in R_+$ $\alpha\in(0,1)$ and $\omega\in\Omega$,where $q:R_+\times\Omega\to C(R_+,R)$, $f,g:R_+\times R\times\Omega\to R$ $\theta,\eta,\mu:R_+\to R_+$ By a random solution of the FNRIE we mean a function $x:\Omega\to BC(R_+,R)$ that satisfies on R_+ . Where $BC(R_+,R)$ is the space of bounded continuous real-valued functions defined on R_+.

(c)Let R denote the real line and R_+ be the set of nonnegative numbers, $R_+=[0,\infty)\subset R$ Let $BM(R_+,R)$ denote the class of bounded continuous real valued functions defined on R_+ Given a measurable space (Ω,A) and for given function $x:\Omega\to BM(R_+,R)$.
Consider the fractional order nonlinear functional random integral equation (in short NFRIE) of mixed type

$$x(t,\omega)=k\left(t,x(\gamma(t),\omega),\omega\right)+\left(\int_0^{\beta(t)} f\left(t,x(\theta(t),\omega),\omega\right)ds\right)\left[q(t,\omega)+\frac{1}{\Gamma(\alpha)}\int_0^{\mu(t)}\frac{g\left(s,x(\eta(s),\omega),\omega\right)}{(t-s)^{1-\alpha}}ds\right]$$

for all $t\in R_+$ $\alpha\in(0,1)$ and $\omega\in\Omega$, where $q:R_+\times\Omega\to R$, $k,f,g:R_+\times R\times\Omega\to R$, $\gamma,\theta,\eta,\beta,\mu:R_+\to R_+$

By a random solution of the NFRIE (1.1) we mean a function $x:\Omega\to BM(R_+,R)$ that satisfies on R_+ . Where $BM(R_+,R)$ is the space of bounded continuous real-valued functions defined on R_+.

(d) we study the existence of locally attractive solutions of the following nonlinear functional integral equation of fractional order (FIEF)

$$x(t)=k\left(t,x(\mu(t))\right)+ f\left(t,x(\gamma(t))\right)\left(q(t)+\frac{1}{\Gamma(\alpha)}\int_0^{\beta(t)}\frac{g\left(s,x(\eta(s))\right)}{(\beta(t)-s)^{1-\alpha}}ds\right) \qquad \forall \quad t\in R_+$$

and α is a fixed number $\alpha\in(0,1)$ Where, $q:R_+\to R$, $f,\mathrm{g},\mathrm{k}:R_+\times R\to R$, and

$\mu,\gamma,\eta,\beta:R_+\to R_+$

By a solution of the FIEF we mean a function $x\in BC(R_+,R)$ that satisfies the equation,

where $BC(R_+,R)$ the space of is all bounded and Continuous real valued functions defined

on R_+

(e) Let R denote the real line, let $I_0 = [-q, 0]$ and $I = [0, T]$, $q, T \geq 0$, be two closed intervals in R_+, and let $J = I_0 \cup I$. Let $C = C(I_0, \mathbb{R})$ be the space of all continuous real-valued functions ϕ on I_0 with the supremum norm $\|\cdot\|_C$ defined by $\|\phi\|_C = \sup_{t \in I_0} |\phi(t)|$.

Clearly, C is Banach algebra with this norm. we let $AC^1(J, R)$ be the space of real-valued continuous functions whose first derivatives exist and are absolutely continuous on J.

We consider the fractional order functional integro-differential equation (FIDE)

$$\left.\begin{aligned} \frac{d^\alpha}{dt^\alpha}\left(\frac{x(t)}{f(t, x(t))}\right) &= \int_0^t g(s, x_s)\,ds, \quad a.e. \ t \in I, \\ x(t) &= \phi(t), \quad t \in I_0, \end{aligned}\right\}$$

where d^α / dt^α denote the Riemann-Liouville derivative of order α, $0 < \alpha < 1$ and the function $x_t : I_0 \to C$ is the continuous function defined by $x_t(\theta) := x(t+\theta)$ for all $\theta \in I_0$ also $f : I \times R \to R \setminus \{0\}$, $g : I \times C \to R$, under suitable mixed Lipschitz and other conditions on the nonlinearities *f* and *g*.

By a solution of the FIDE, we mean a function $x \in AC^1(J, R)$ such that:

(i) the function $t \to \left(\dfrac{x}{f(t, x)}\right)$ is absolutely continuous for each $x \in R$;

(ii) x Satisfies equation.

While fractional order functional differential equations have been a very active area of research for a long time, the study of fractional order functional integro-differential equations in Banach algebras is relatively new to the literature.

8. Conclusion: The present analysis exhibits the applicability of the fixed point method to exist the solution for nonlinear integral and differential equations of fractional order. The fixed point method is the most reliable method because of its simplicity. We also show that solution of these equations is locally attractive. The conclusions for separate equations are described in the research papers.

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